Physics 523, Quantum Field Theory II
Homework 2
Due Wednesday, 21st January 2004

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1. Feynman Parametrization

We are to prove *Feynman's Formula*,

$$\frac{1}{A_1\cdots A_n} = \int_0^1 dx_1\cdots dx_n \delta^{(n)}\left(\sum_{i=1}^n x_i - 1\right)\frac{(n-1)!}{[x_1A_1+\cdots+x_nA_n]^n}. \tag{1.1}$$

We will prove this result by induction. First, we will show that

$$\frac{1}{A_1A_2} = \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{1}{[x_1A_1+x_2A_2]^2}. \tag{1.2}$$

This integral can be simplified by using the dirac δ-function so that,

$$\int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{1}{[x_1A_1+x_2A_2]^2} = \int_0^1 dx_1\frac{1}{[x_1A_1+(1-x_1)A_2]^2}. \tag{1.3}$$

We will solve this integral by making the substitution $u \equiv (x_1A_1+(1-x_1)A_2)$ so that $du = (A_1-A_2)dx_1$. Substituting u in the integral above and noting the change in the limits of integration we see immediately that

$$\frac{1}{A_1A_2} = \int_{A_2}^{A_1}\frac{du}{(A_1-A_2)}\frac{1}{u^2} = \frac{1}{(A_1-A_2)}\left(-\frac{1}{u}\right)\Bigg|_{A_2}^{A_1} = \frac{1}{A_1-A_2}\left(\frac{1}{A_2}-\frac{1}{A_1}\right), \tag{1.4}$$

$$\therefore \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{1}{[x_1A_1+x_2A_2]^2} = \frac{1}{A_1A_2}. \tag{1.5}$$

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Before we complete our proof, let us prove the lemma,

$$\frac{1}{A_1A_2^n} = \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{nx_2^{n-1}}{[x_aA_1+x_2A_2]^{n+1}}. \tag{1.6}$$

This lemma will be proved by induction. We have shown that for $n=1$ equation (1.6) holds. Now, let us suppose that (1.6) is true for some exponent $m \geq 1$. We must show that this implies that (1.6) is satisfied for $m+1$. So our induction hypothesis is given by

$$\frac{1}{A_1A_2^m} = \int_0^1 dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{mx_2^{m-1}}{[x_aA_1+x_2A_2]^{m+1}}. \tag{1.7}$$

Let us differentiate both side of equation (1.7) with respect to A_2. This becomes

$$-m\frac{1}{A_1A_2^{m+1}} = -\int dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{m(m+1)x_2^{m-1}x_2}{[x_1A_1+x_2A_2]^{m+2}}, \tag{1.8}$$

$$\therefore \frac{1}{A_1A_2^{m+1}} = \int dx_1dx_2\delta^{(2)}(x_1+x_2-1)\frac{(m+1)x_2^m}{[x_1A_1+x_2A_2]^{m+2}}. \tag{1.9}$$

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Now we are ready to complete the entire proof. Because we have shown that Feynman's formula is true for $\frac{1}{A_1A_2}$, we may prove by induction to $\frac{1}{A_1\cdots A_n}$. Let us assume therefore that Feynman's formula is valid for some $n = m \geq 2$. We must show that it is valid for $m+1$.

We will begin this proof by direct calculation. For this derivation, we will use the following notational conveniences:

$$\mathscr{U} \equiv (x_1A_1 + \cdots + x_nA_m), \quad u_i \equiv (1-u_{m+1})x_i, \quad du_i \equiv (1-x_{m+1})dx_i \text{ for } i \in [1,m].$$

Note that u_{m+1} is an ordinary integration variable and is not set by the above. By our induction hypothesis, we have that

$$\frac{1}{A_1\cdots A_m} = \int_0^1 dx_1\cdots dx_m \delta^{(m)}\left(\sum_{i=1}^m x_i - 1\right)\frac{(m-1)!}{[x_1A_1+\cdots+x_mA_m]^m}.$$

We also note the property of the Dirac δ-functional that $\delta(f(x)/a) = a\delta(f(x))$. Now, let us make the following calculation

$$\begin{aligned}
\frac{1}{A_1\cdots A_{m+1}} &= \frac{1}{A_{m+1}}\frac{1}{A_1\cdots A_m},\\
&= \frac{1}{A_{m+1}}\int_0^1 dx_1\cdots dx_m\delta\left(\sum_{i=1}^m x_i - 1\right)\frac{(m-1)!}{[x_1A_1+\cdots+x_mA_m]^m},\\
&= \int_0^1 dx_1\cdots dx_m\delta\left(\sum_{i=1}^m x_i - 1\right)(m-1)!\frac{1}{\mathscr{U}^m}\frac{1}{A_{m+1}},\\
&= \int_0^1 dx_1\cdots dx_m\delta\left(\sum_{i=1}^m x_i - 1\right)(m-1)!\int_0^1 du_{m+1}\frac{m(1-u_{m+1})^{m-1}}{[(1-u_{m-1})\mathscr{U} + u_{m+1}A_{m+1}]^{m+1}},\\
&= \int_0^{1-u_{m+1}} du_1\cdots\ du_m\delta\left(\sum_{i=1}^m x_i - 1\right)\frac{m!}{(1-x_{m+1})^m}\int_0^1 du_{m+1}\frac{(1-u_{m+1})^{m-1}}{[u_1A_1+\cdots+u_mA_m+u_{m+1}A_{m+1}]^{m+1}},\\
&= \int_0^1 du_{m+1}\int_0^{1-u_{m+1}} du_1\cdots du_m\delta\left(\sum_{i=1}^m \frac{u_i}{(1-u_{m+1})} - 1\right)\frac{m!}{(1-u_{m+1})[u_1A_1+\cdots+u_{m+1}A_{m+1}]^{m+1}},\\
&= \int_0^1 du_{m+1}\int_0^{1-u_{m+1}} du_1\cdots du_m\delta\left(\sum_{i=1}^{m+1} u_i - 1\right)\frac{m!}{[u_1A_1+\cdots+u_{m+1}A_{m+1}]^{m+1}}.
\end{aligned}$$

We note that because of the δ-functional within the integral (and because u_{m+1} is always positive), when the domain of the interior integral is extended to 1 the integral will not pick up any additional contribution. So we may put the integral above into a more symmetric form,

$$\frac{1}{A_1\cdots A_{m+1}} = \int_0^1 du_1\cdots du_{m+1}\delta\left(\sum_{i=1}^{m+1} u_i - 1\right)\frac{m!}{[u_1A_1+\cdots+u_{m+1}A_{m+1}]^{m+1}}. \tag{1.10}$$

Therefore, by induction on m we see that for all values $n \geq 2$,

$$\frac{1}{A_1\cdots A_n} = \int_0^1 dx_1\cdots dx_n\delta^{(n)}\left(\sum_{i=1}^n x_i - 1\right)\frac{(n-1)!}{[x_1A_1+\cdots+x_nA_n]^n}. \tag{1.11}$$

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2. Loop Integrals

a) We are to demonstrate that

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}} \quad \text{for } m > 2.$$

To compute this integral, we will first note that the two poles, at $\ell = \pm\sqrt{\Delta}$, are covered by the same contour in the complex ℓ^0 plane when the contour is analytically extended to the imaginary axis. Therefore, without loss of generality, we may make the substitution $\ell = i\ell_E$.

Doing this, we may compute directly. Note the substitution $u \equiv \ell_E^2 + \Delta$ in the fifth line. Also notice that the derivation is only valid for $m > 2$ because the integral will diverge for $m \leq 2$.

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} &= \frac{i}{(2\pi)^4}\int d^4\ell_E \frac{1}{[-\ell_E^2-\Delta]^m},\\
&= \frac{i(-1)^m}{(2\pi)^4}\int d^4\ell_E \frac{1}{[\ell_E^2+\Delta]^m},\\
&= \frac{i(-1)^m}{(2\pi)^4}\int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^m},\\
&= \frac{2i(-1)^m}{(4\pi)^2}\int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^m},\\
&= \frac{2i(-1)^m}{(4\pi)^2}\int_\Delta^\infty \frac{du}{2\ell_E}\frac{\ell_E^3}{u^m},\\
&= \frac{i(-1)^m}{(4\pi)^2}\int_\Delta^\infty du \frac{u-\Delta}{u^m},\\
&= \frac{i(-1)^m}{(4\pi)^2}\left(\frac{1}{(m-1)}\frac{\Delta}{u^{m-1}} - \frac{1}{(m-2)}\frac{1}{u^{m-2}}\right)\Bigg|_\Delta^\infty,\\
&= \frac{i(-1)^m}{(4\pi)^2}\left(\frac{1}{(m-2)}\frac{1}{\Delta^{m-2}} - \frac{1}{(m-1)}\frac{1}{\Delta^{m-2}}\right),
\end{aligned}$$

$$\therefore \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}} \quad \text{for } m > 2. \tag{2.1}$$

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b) We are to demonstrate that

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{[\ell^2-\Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}} \quad \text{for } m > 3.$$

To prove this equality we will proceed similarly to part (a) above. Like before, we note that the two residues, at $\ell = \pm\sqrt{\Delta}$, are covered by the same branch cut in the complex plane when the contour integral is analytically continued to the imaginary axis. Therefore, we will make the substitution $\ell = i\ell_E$. When computing the integral explicitly below, note the substitution $u \equiv \ell_E^2 + \Delta$. Also, notice that for $m \leq 3$ the integral will diverge. We will proceed directly.

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{[\ell^2-\Delta]^m} &= \frac{i}{(2\pi)^4}\int d^4\ell_E \frac{-\ell_E^2}{[-\ell_E^2-\Delta]^m},\\
&= \frac{i(-1)^{m-1}}{(2\pi)^4}\int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^5}{[\ell_E^2+\Delta]^m},\\
&= \frac{2i(-1)^{m-1}}{(4\pi)^2}\int_\Delta^\infty \frac{du}{2\ell_E}\frac{\ell_E^5}{u^m},\\
&= \frac{i(-1)^{m-1}}{(4\pi)^2}\int_\Delta^\infty du \frac{\ell_E^4}{u^m},\\
&= \frac{i(-1)^{m-1}}{(4\pi)^2}\int_\Delta^\infty du \frac{(u^2-2\Delta u+\Delta^2)}{u^m},\\
&= \frac{i(-1)^{m-1}}{(4\pi)^2}\left(-\frac{1}{(m-3)}\frac{1}{u^{m-3}} + \frac{1}{(m-2)}\frac{2\Delta}{u^{m-2}} - \frac{1}{(m-1)}\frac{\Delta^2}{u^{m-1}}\right)\Bigg|_\Delta^\infty,\\
&= \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{1}{\Delta^{m-3}}\left(\frac{1}{(m-3)} - \frac{2}{(m-2)} + \frac{1}{(m-1)}\right),
\end{aligned}$$

$$\therefore \int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{[\ell^2-\Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}} \quad \text{for } m > 3. \tag{2.2}$$

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c) Let us prove the identity

$$\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)=\frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right).$$

To prove this identity we will differentiate both sides with respect to Δ. Doing this, the right hand side trivially becomes (noting the definition of Δ_Λ),

$$\frac{\partial}{\partial\Delta}\left\{\frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right)\right\}=\frac{i}{(4\pi)^2}\frac{\Delta}{\Delta_\Lambda}\left(-\frac{z\Lambda^2}{\Delta^2}\right)=\frac{i}{(4\pi)^2}\frac{-z\Lambda^2}{\Delta_\Lambda\Delta}. \tag{2.3}$$

Differentiating the left hand side and using equation (2.2) we see that,

$$\begin{aligned}\frac{\partial}{\partial\Delta}\left\{\int\frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)\right\}&=3\int\frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^4}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^4}\right),\\&=3\frac{i(-1)^3\cdot 2}{(4\pi)^2(3\cdot2\cdot1)}\left(\frac{1}{\Delta}-\frac{1}{\Delta_\Lambda}\right),\\&=\frac{i}{(4\pi)^2}\left(\frac{1}{\Delta_\Lambda}-\frac{1}{\Delta}\right),\\&=\frac{i}{(4\pi)^2}\left(\frac{\Delta-\Delta_\Lambda}{\Delta_\Lambda\Delta}\right),\\&=\frac{i}{(4\pi)^2}\frac{-z\Lambda^2}{\Delta_\Lambda\Delta},\end{aligned}$$

Therefore the derivatives of each sides of the desired identity with respect to Δ are equal. We note that, by direct calculation, the constant of integration is zero.

$$\therefore\int\frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{[\ell^2-\Delta]^3}-\frac{\ell^2}{[\ell^2-\Delta_\Lambda]^3}\right)=\frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right). \tag{2.4}$$

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3. The Volume Element in D-Dimensions

a) We note that evaluating the trivial Gaussian integral yields

$$I=\int_{-\infty}^{\infty}dx\ e^{-x^2}=\sqrt{\pi}. \tag{3.1}$$

b) Let us compute the general Gaussian integral,

$$I^n=\int_{-\infty}^{\infty}dx_1\cdots dx_n\ e^{-(x_1^2+\cdots+x_n^2)}.$$

We note that a general procedure for computing such Gaussian integrals is to convert it into an integral over spherical coordinates. Let us compute I^n directly this way. When needed, we will define the substitution variable $u\equiv r^2$.

$$\begin{aligned}I^n&=\int d\Omega_{n-1}\int_0^\infty dr\ r^{n-1}e^{-r^2},\\&=\int d\Omega_{n-1}\int_0^\infty\frac{du}{2r}\ r^{n-1}e^{-u},\\&=\int d\Omega_{n-1}\frac{1}{2}\int_0^\infty du\ u^{(n-2)/2}e^{-u},\\&=\frac{1}{2}\Gamma(n/2)\Omega_{n-1},\end{aligned}$$

c) Using our result above we see that $\pi^{(D/2)}=\Omega_{D-1}1/2\Gamma(D/2)$. Therefore it is clear that

$$\Omega_{D-1}=\frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{3.2}$$

d) Therefore by part (c) we see immediately that

$$\Omega_1=2\pi,\quad\Omega_2=4\pi,\quad\Omega_3=2\pi^2,\quad\Omega_4=\frac{8}{3}\pi^2. \tag{3.3}$$

4. The Electron Vertex Function

We are to completely simplify the numerator of the integrand of the electron vertex function's first order correction written as

$$\mathscr{N} \equiv \overline{u}(p')\left[\, \not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right] u(p). \tag{4.1}$$

In part to accomplish this task we will make the substitution

$$\ell \equiv k + yq - zp.$$

During the following exercise in algebra, we will often make use of the Dirac equation which can be written as

$$\overline{u}(p')\not{p}' = \overline{u}(p')m, \qquad \not{p}u(p) = mu(p), \qquad \overline{u}(p')\not{q}u(p) = 0, \tag{4.2}$$

and we will frequently imply the use of the Dirac equation to set $\not{p} \to m, \not{p}' \to m$, or $\not{q} \to 0$ by implying contraction with a spinor outside the square brackets. This of course can only be done when the specific momentum 4-vector is appropriately located (without γ^μ's between it and the needed spinor(s)). Also, we will make use of the facts derived in class that when this integral is evaluated, all terms linear in ℓ^ν will give no contribution and rotational symmetry allows us to set $\ell^\mu\ell^\mu \to \frac{1}{4}g^{\mu\nu}\ell^2$.[1]

Let us begin our calculation by direct substitution (making use of the stated identity to throw out terms linear in ℓ^ν).

$$\mathscr{N} = \overline{u}(p')\left[\underbrace{\not{\ell}\gamma^\mu \not{\ell}}_{\textbf{i}} - y(1-y)\underbrace{\not{q}\gamma^\mu \not{q}}_{\textbf{ii}} - zy\underbrace{\not{q}\gamma^\mu \not{p}}_{\textbf{iii}} + z(1-y)\underbrace{\not{p}\gamma^\mu \not{q}}_{\textbf{iv}} + z^2\underbrace{\not{p}\gamma^\mu \not{p}}_{\textbf{v}} + m^2\gamma^\mu - 2m(1-2y)q - 4mzp\right] u(p).$$

We will evaluate this in parts.

i. [2]
$$\not{\ell}\gamma^\mu \not{\ell} = 2\not{\ell}\ell - \not{\ell}^{\,2}\gamma^\mu = \frac{1}{2}g^{\mu\nu}\gamma_\nu\ell^2 - \ell^2\gamma^\mu = -\frac{1}{2}\ell^2\gamma^\mu.$$

ii.
$$\not{q}\gamma^\mu \not{q} = \underbrace{2\not{q}q}_{\to 0} - \not{q}^2\gamma^\mu = -q^2\gamma^\mu.$$

iii.
$$\not{q}\gamma^\mu \not{p} = \not{q}\gamma^\mu m = m\not{p}'\gamma^\mu - m\not{p}\gamma^\mu = m^2\gamma^\mu - 2mp^\mu + m^2\gamma^\mu = 2m^2\gamma^\mu - 2mp^\mu.$$

iv.
$$\not{p}\gamma^\mu \not{q} = \underbrace{2p\not{q}}_{\to 0} - \gamma^\mu\not{p}\not{q} = -2\gamma^\mu p\cdot q + m\gamma^\mu\not{q} = -\gamma^\mu 2p\cdot q + m\gamma^\mu\not{p}' - m^2\gamma^\mu,$$
$$= -\gamma^\mu 2p\cdot q + 2mp'^\mu - 2m^2\gamma^\mu.$$

Notice, however, that

$$2p\cdot q = p\cdot q + p\cdot q = p\cdot q + p'\cdot q - q^2 = p'^2 + p'\cdot p - p'\cdot p - p^2 - q^2 = m^2 - m^2 - q^2 = -q^2.$$

Therefore,

$$\not{p}\gamma^\mu \not{q} = \gamma^\mu q^2 + mp'^\mu - 2m^2\gamma^\mu.$$

v.
$$\not{p}\gamma^\mu \not{p} = m\not{p}\gamma^\mu = 2mp^\mu - m^2\gamma^\mu.$$

Combining all of these results, we may write the numerator as

$$\mathscr{N} = \overline{u}(p')\left[\gamma^\mu\overbrace{\left(-\frac{1}{2}\ell^2 + y(1-y)q^2 + z(1-y)q^2 - 2m^2yz - 2m^2z(1-y) - z^2m^2 + m^2\right)}^{\mathscr{A}},\right.$$
$$\left. + \underbrace{2myzp^\mu + 2mz(1-y)p'^\mu + 2mz^2p^\mu - 2m(1-2y)q^\mu - 4mzp^\mu}_{\mathscr{B}}\right] u(p).$$

[1] It is important to note that we do *not* imply that $\ell^\mu\ell^\nu = \frac{1}{4}g^{\mu\nu}\ell^2$ or that $\ell^\nu = 0$ but rather that these are symmetries of the integrand.

[2] Here and later in the derivation we make use of the identity $\not{p}^2 = p^2$. This is seen by simple γ algebra: $\not{p}^2 = p_\nu\gamma^\nu\gamma^\mu p_\mu = 2p^2 - p_\mu\gamma^\mu\gamma^\nu p_\nu = 2p^2 - \not{p}^2$. So $2\not{p}^2 = 2p^2 \Longrightarrow \not{p}^2 = p^2$.

Let us simplify the parts $\mathscr{A}$ and $\mathscr{B}$ separately. To do this, we will make repeated use of the fact that $x+y+z=1$ by the Dirac δ-functional of these Feynman parameters. Let us begin with part $\mathscr{A}$.

$$\begin{aligned}\mathscr{A} &= -\frac{1}{2}\ell^2 + q^2\left(y(1-y)+z(1-y)\right) + m^2\left(-2yz-2z(1-y)-z^2+1\right),\\ &= -\frac{1}{2}\ell^2 + q^2\left((1-x-z)(1-z)+z(1-y)\right) + m^2\left(-2yz-2z+2yz+1\right),\\ &= -\frac{1}{2}\ell^2 + q^2(1-x)(1-y) + m^2(1-2z-z^2).\end{aligned}$$

Now let us simplify part $\mathscr{B}$. This process will not seem beautiful or elegant, but in the words of Pascal, "I apologize for this [derivation's] length for I did not have time to make it short."

$$\begin{aligned}\mathscr{B} &= 2myzp^\mu + 2mz(1-y)p'^\mu + 2mz^2p^\mu - 2m(1-2y)q^\mu - 4mzp^\mu,\\ &= 2m\left(yzp^\mu + zp'^\mu - zyp'^\mu + z^2p^\mu - q^\mu + 2yq^\mu - 2zp^\mu\right),\\ &= 2m\left(z(z-1)p^\mu - zp^\mu + zp'^\mu - zp'^\mu + zxp'^\mu + z^2p'^\mu - q^\mu + 2yq^\mu + yzp^\mu\right),\\ &= 2m\left(z(z-1)(p^\mu+p'^\mu) + zq^\mu + zxp'^\mu - q^\mu + 2yq^\mu + yzp^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) + z^2p^\mu - zp^\mu + z^2p'^\mu - zp'^\mu + 2zp'^\mu - 2zyp'^\mu - 2z^2p'^\mu\right.\\ &\qquad\qquad \left. + 2zp^\mu - 2xzp^\mu - 2z^2p'^\mu + 4yq^\mu + 2zq^\mu - 2q^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) - z^2p^\mu + zp^\mu - z^2p'^\mu + zp'^\mu - 2zyp'^\mu - 2xzp^\mu + 4yq^\mu + 2zq^\mu - 2q^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) - zp^\mu + zxp^\mu + zyp^\mu + zp^\mu \;\; - zp'^\mu + zxp'^\mu + ztp'^\mu + zp'^\mu - 2zyp'^\mu\right.\\ &\qquad\qquad \left. -2zxp^\mu + 4yp^\mu + 2zp'^\mu - 2zp^\mu - 2p'^\mu + 2p^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) - zyp'^\mu + zxp'^\mu - zxp^\mu + zyp^\mu + 2yp'^\mu - 2yp^\mu + 2p'^\mu - 2xp'^\mu - 2zp'^\mu\right.\\ &\qquad\qquad \left. -2p^\mu + 2xp^\mu + 2zp^\mu + 2zp'^\mu - 2zp^\mu - 2p'^\mu + 2p^\mu\right),\\ &= m\left(z(z-1)(p^\mu+p'^\mu) + (p'^\mu - p^\mu)(zx - zy + 2y - 2x)\right),\\ &= mz(z-1)(p^\mu+p'^\mu) + mq^\mu(z-2)(x-y).\end{aligned}$$

When we combine these simplifications into the entire expression for the numerator, we see that

$$\therefore \mathscr{N} = \overline{u}(p')\left[\gamma^\mu\left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + m^2(1-2z-z^2)\right) + mz(z-1)\left(p'^\mu + p^\mu\right) + m(z-2)(x-y)q^\mu\right].$$

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